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WHEN THE TRIVIAL IS NONTRIVIAL

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Any abelian group $(G, +)$ with identity 0 can be made into a ring by defining the trivial multiplication $a * b = 0$. Surprisingly, there are nontrivial abelian groups for which this is the only possible way to define a ring. We start by showing that $\mathbb{Q}/\mathbb{Z}$, the quotient group of the rationals modulo the integers, is such a group. We find other such groups and search for criteria characterizing groups that can have no nontrivial multiplication. This paper grew out of the first author's undergraduate project for an abstract algebra course.

Example 1. Let $(\mathbb{Q}, +)$ be the rationals under addition and $(\mathbb{Z}, +)$ be the subgroup of integers. We can think of the quotient group $\mathbb{Q}/\mathbb{Z}$ as rational points around a circle, as in Figure 1. Any $\frac{a}{b} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ with $b > 0$ has order b, provided $\frac{a}{b}$ is in reduced

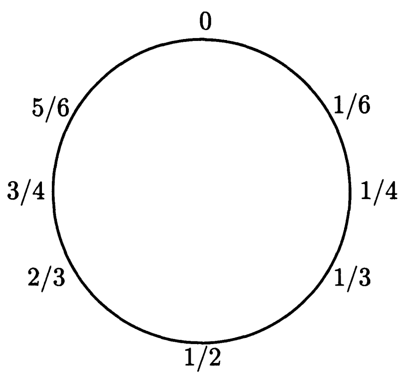

FIG. 1. *Rational points around a circle.*

form. That is, with the usual notation for repeated addition, $b(\frac{a}{b} + \mathbb{Z}) = 0 + \mathbb{Z}$. For simplicity, we'll write $\frac{a}{b}+\mathbb{Z}$ as $\left[\frac{a}{b}\right]$. We show that distributivity completely limits what multiplication can be defined on $\mathbb{Q}/\mathbb{Z}$. Let $*$ be any multiplication that makes $\mathbb{Q}/\mathbb{Z}$ a ring. As before, for $\left[\frac{a}{b}\right] \in \mathbb{Q}/\mathbb{Z}$ with $b \in \mathbb{N}$, $b(\left[\frac{a}{b}\right]) = 0$. Also, for any $\left[\frac{c}{d}\right] \in \mathbb{Q}/\mathbb{Z}$, we have $\left[\frac{c}{d}\right] = b(\left[\frac{c}{bd}\right])$. Then, using distributivity,

$$\left[\frac{a}{b}\right] * \left[\frac{c}{d}\right] = \left[\frac{a}{b}\right] * b\left(\left[\frac{c}{bd}\right]\right) = b\left(\left[\frac{a}{b}\right] * \left[\frac{c}{bd}\right]\right) = \left(b\left[\frac{a}{b}\right]\right) * \left[\frac{c}{d}\right] = 0.$$

The preceding argument used two properties of $\mathbb{Q}/\mathbb{Z}$. First of all, every element is of finite order. Secondly, for every $n \in \mathbb{N}$ and every element x, there is an element y such that adding y to itself n times gives $n(y) = x$. These properties appear sufficiently often in the study of infinite groups to merit the definitions below. The third definition replaces the cumbersome phrase about rings whose only multiplication is the trivial one. (See [2], pp. 15, 58, 272.)

Definition. A group is *torsion* iff every element is of finite order.

Definition. A group $(G, +)$ is *divisible* iff for all $x \in G$ and for all $n \in \mathbb{N}$, there is $y \in G$ such that $n(y) = x$.

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Definition. An abelian group G with identity 0 is *nil* iff the only multiplication on G that makes G into a ring is defined by for all $a, b \in G$, $a * b = 0$, the trivial multiplication.

Let's consider familiar groups in light of these definitions. The group of integers lacks both torsion and divisibility and is definitely not nil since the usual multiplication is nontrivial. The rationals and reals are divisible, using $y = \frac{x}{n}$. However, they have elements of infinite order and of course have familiar nontrivial multiplications. Finite abelian groups are torsion groups, but with the exception of the group $\mathbb{Z}_1$, they can't be divisible. And, as we'll see after Theorem 1, every nontrivial finite abelian group can have a nontrivial multiplication. So neither of the conditions torsion or divisibility are sufficient separately to force a group to be nil. However, a direct modification of the argument in Example 1 shows that together these two properties suffice. We will later consider the necessity of these two properties.

THEOREM 1. *If G is a divisible torsion abelian group, then G is nil.*

Proof. Let $g, h \in G$, where $(G, +)$ is a divisible abelian group and every element is of finite order. Let $*$ be any binary operation such that $(G, +, *)$ is a ring. Suppose the order of g is n. By divisibility of G there is $j \in G$ such that $nj = h$. By distributivity and elementary ring properties $g * h = g * (nj) = n(g * j) = (ng) * j = 0 * j = 0$. Hence G is nil. □

The group with one element is nil. However, all other finite abelian groups have nontrivial multiplications: Let G be any finite abelian group with more than one element. By the Fundamental Theorem of Finite Abelian Groups, G is the direct product of cyclic groups with more than one element. (See [3], p. 217.) Turn each cyclic group with n elements into the ring $\mathbb{Z}_n$. Then the direct product of these rings has a nontrivial multiplication and the additive group is isomorphic to G.

Two initial questions arise naturally: Are there groups besides $\mathbb{Q}/\mathbb{Z}$ that share the hypotheses of Theorem 1? Are these properties necessary for all nil groups? Example 2 and Theorem 2 answer the first question affirmatively, while Example 4 gives a negative answer to the second question.

Example 2. Let

$$\mathbb{Q}_2/\mathbb{Z} = \{ \left[\frac{a}{2^i}\right] : i \in \mathbb{N},\ 0 \le a < 2^i\}.$$

This is a subgroup of $\mathbb{Q}/\mathbb{Z}$, so every element is of finite order. We show it is divisible. Let $\left[\frac{a}{2^i}\right] \in \mathbb{Q}_2/\mathbb{Z}$ and $n \in \mathbb{N}$. Then we can write $n = 2^k j$, where j is an odd positive integer and k is a nonnegative integer. First of all, $2^k(\left[\frac{a}{2^i 2^k}\right]) = \left[\frac{a}{2^i}\right]$. Now j is relatively prime to 2^{i+k} and $H = \{\left[\frac{a}{2^{i+k}}\right] :\ 0 \le a < 2^{i+k}\}$ is a finite cyclic group with 2^{i+k} elements. Hence, the mapping $\phi : H \to H$ given by $\phi(\left[\frac{a}{2^{i+k}}\right]) = \left[\frac{aj}{2^{i+k}}\right]$ is an automorphism of H. (See [3], p. 131.) Thus there is some $\left[\frac{b}{2^{i+k}}\right] \in H$ such that $j(\left[\frac{b}{2^{i+k}}\right]) = \left[\frac{a}{2^{i+k}}\right]$. Hence,

$$n(\left[\frac{b}{2^{i+k}}\right]) = 2^k j(\left[\frac{b}{2^{i+k}}\right]) = 2^k(\left[\frac{a}{2^{i+k}}\right]) = \left[\frac{a}{2^i}\right].$$

By Theorem 1, $\mathbb{Q}_2/\mathbb{Z}$ is nil.

We can modify Example 2 to permit any set of primes to appear as factors in the denominator, as long as any primes that appear in the denominator can be raised to any positive exponent. That is, suppose P is any nonempty set of prime numbers, finite or infinite, and let $\mathbb{Q}_P/\mathbb{Z} = \{\left[\frac{a}{b}\right] : 0 \le a < b$ and there are finitely many primes

$p_i \in P$ and nonnegative integers k_i such that $b = \prod_{i=1}^{n} p_i^{k_i}\}$. Then $\mathbb{Q}_P/\mathbb{Z}$ is a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$. (Different authors use different notations and names for the groups $\mathbb{Q}_q/\mathbb{Z}$, for q a prime, including quasicyclic and *q-primary component*. See [2], pp. 23 - 24.)

The next example illustrates that we need unlimited exponents for divisible subgroups of $\mathbb{Q}/\mathbb{Z}$.

Example 3. Let $\mathbb{Q}_{SF}/\mathbb{Z} = \{\left[\frac{a}{b}\right] : 0 \leq a < b$ and b is a square free integer$\}$. Again, this set forms an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$ and all elements have finite order. The square free condition means that no prime can have a higher power than 1 in the denominator. This means $\mathbb{Q}_{SF}/\mathbb{Z}$ is not divisible: 2 is in $\mathbb{N}$ and $\frac{1}{2} \in \mathbb{Q}_{SF}/\mathbb{Z}$, but the only solutions in $\mathbb{Q}/\mathbb{Z}$ to $2(\left[\frac{a}{b}\right]) = \left[\frac{1}{2}\right]$ are $\left[\frac{1}{4}\right]$ and $\left[\frac{3}{4}\right]$, neither of which are in $\mathbb{Q}_{SF}/\mathbb{Z}$.

Furthermore, we can define nontrivial multiplications on $\mathbb{Q}_{SF}/\mathbb{Z}$. For example, given reduced fractions define

$$\left[\frac{a}{b}\right] * \left[\frac{c}{d}\right] = \begin{cases} \left[\frac{1}{2}\right] & \text{if both } b \text{ and } d \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

We prove that $(\mathbb{Q}_{SF}/\mathbb{Z}, +, *)$ is a ring. For associativity, note that the product of three fractions is 0, unless all three denominators are even, in which case the product is $\left[\frac{1}{2}\right]$. For distributivity we need to consider $(\left[\frac{a}{b}\right] * \left[\frac{c}{d}\right]) + (\left[\frac{a}{b}\right] * \left[\frac{e}{f}\right])$ and $\left[\frac{a}{b}\right] * (\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right])$. When b is odd, both terms are 0. Thus we suppose b is even. If both d and f are odd, the denominator of $\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]$ is also odd and so both terms are again 0. Next let, say, d be odd and f be even. Then $(\left[\frac{a}{b}\right] * \left[\frac{c}{d}\right]) + (\left[\frac{a}{b}\right] * \left[\frac{e}{f}\right]) = [0] + \left[\frac{1}{2}\right] = \left[\frac{1}{2}\right]$. Also, e must be odd for $\left[\frac{e}{f}\right]$ to be reduced. Hence $\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]$ will have an odd numerator and even denominator. This means that $\left[\frac{a}{b}\right] * (\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]) = \left[\frac{1}{2}\right]$ as well. Finally, suppose that both d and f are even. Then $(\left[\frac{a}{b}\right] * \left[\frac{c}{d}\right]) + (\left[\frac{a}{b}\right] * \left[\frac{e}{f}\right]) = \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] = [0]$. Furthermore, c and e are odd since the fractions are reduced. Because both d and f have exactly one factor of 2 in them, the common denominator for $\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]$ has just one factor of 2. Hence in adding them, each denominator will be multiplied by an odd number to get the common denominator. Thus the two new numerators will be odd and their sum even. Then the reduced sum has an odd denominator and $\left[\frac{a}{b}\right] * (\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]) = [0]$. This finishes showing distributivity and so $(\mathbb{Q}_{SF}/\mathbb{Z}, +, *)$ is a ring with nontrivial multiplication.

The next theorem uses familiar algebra constructions to find other nil groups that are somewhat more general than subgroups of $\mathbb{Q}/\mathbb{Z}$.

THEOREM 2. *The direct product of two divisible torsion abelian groups is a divisible torsion abelian groups. Similarly, the homomorphic image of a divisible torsion abelian group is a divisible torsion abelian group. Thus these direct products and homomorphic images are also nil groups.*

Proof. Suppose G and H are divisible torsion abelian groups. Let $(g, h) \in G \times H$, their direct product, which is abelian. The order of (g, h) is the least common multiple of the orders of g and h and so is finite. For $n \in \mathbb{N}$, by the divisibility of G and H, there are elements $g' \in G$ and $h' \in H$ such that $ng' = g$ and $nh' = h$. Then $n(g', h') = (g, h)$, showing $G \times H$ is divisible. By Theorem 1, $G \times H$ is nil.

Now suppose that $\phi : G \to K$ is a homomorphism from G onto a group K. Since G is abelian and ϕ is onto, K is abelian. Let $k \in K$. Since ϕ is onto, there is $g \in G$ such that $\phi(g) = k$. The order of g is finite, say n. Then $nk = n\phi(g) = \phi(ng) = \phi(0) = 0$. Thus k has finite order. Now let $s \in \mathbb{N}$. By divisibility of G, there is some $g' \in G$

such that $sg' = g$. Then $s\phi(g') = \phi(sg') = \phi(g) = k$. So $\phi(g')$ fulfills the definition for K to be divisible. Thus K is nil. □

Surprisingly neither of the conditions torsion or divisible is necessary, as Example 4 shows:

Example 4. Let $\mathbb{Q}_{SF} = \{\frac{a}{b} : \ b \text{ is square free}\}$. Note that this group is not divisible since, for example, $\frac{1}{2}$ can't be "divided in half": In $\mathbb{Q}$, if $x + x = \frac{1}{2}$, then $x = \frac{1}{4}$, which is not in $\mathbb{Q}_{SF}$. Also, every nonzero element of $\mathbb{Q}_{SF}$ is of infinite order. Nevertheless, the only multiplication on $\mathbb{Q}_{SF}$ is the trivial multiplication.

Proof. Suppose $*$ is a binary operation that makes $(\mathbb{Q}_{SF}, +, *)$ a ring. For a contradiction, suppose that for some nonzero $\frac{a}{b}$ and $\frac{c}{d}$ in $\mathbb{Q}_{SF}$ their product $\frac{a}{b} * \frac{c}{d} = \frac{s}{t}$ is nonzero. Then by distributivity $\frac{s}{t} = \frac{a}{b} * \frac{c}{d} = a(\frac{1}{b}) * c(\frac{1}{d}) = ac(\frac{1}{b} * \frac{1}{d})$. Thus $\frac{1}{b} * \frac{1}{d} = \frac{s}{act}$ and so $1 * 1 = \frac{bds}{act}$, which isn't 0. For ease, write $1 * 1 = \frac{x}{y}$ in reduced terms. For every prime p, $\frac{x}{y} = 1 * 1 = (p\frac{1}{p}) * (p\frac{1}{p}) = p^2(\frac{1}{p} * \frac{1}{p})$ and so $\frac{1}{p} * \frac{1}{p}$ is not zero. Furthermore since $\frac{1}{p} * \frac{1}{p}$ is in $\mathbb{Q}_{SF}$, it has at most one factor of p in its denominator. Thus x must have a factor of p in it. However, this would be true for every prime p, which would make x infinite, a contradiction. Hence, $\mathbb{Q}_{SF}$ is nil. □

Since the group of Example 3 is the homomorphic image of the group in Example 4, the homomorphic image of a nil group need not be nil. We can use an advanced theorem to show that the direct product of nil groups need not be a nil group. A *mixed* group contains both elements of infinite order and non-identity elements of finite order. For example, $\mathbb{Q}_{SF} \times (\mathbb{Q}_2/\mathbb{Z})$ is mixed. Theorem 71.1 from [2], p. 272, shows that no mixed group is nil. It also shows a partial converse to Theorem 1: If a torsion group is nil, then it is divisible.

It appears difficult to characterize nil groups completely because of the variety of "torsion free" groups—groups whose only element of finite order is the identity. (See [2], p. 272.) We saw in Theorem 1 that divisible torsion abelian groups are nil, but groups, such as $\mathbb{Q}$ or $\mathbb{Z}_5$, lacking either divisible or torsion can fail to be nil. However, it is possible, as in Example 4, for a group to lack both torsion and divisibility and still be nil.

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